Preferential concentration versus clustering in inertial particle transport by random velocity fields

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The concept of preferential concentration in the transport of inertial particles by random velocity fields is extended to account for the possibility of zero correlation time and compressibility of the velocity field. It is shown that, in the case of an uncorrelated in time random velocity field, preferential concentration takes the form of a condition on the field history leading to the current particle positions. This generalized form of preferential concentration appears to be a necessary condition for clustering in the uncorrelated in time case. The standard interpretation of preferential concentration is recovered considering local time averages of the velocity field. In the compressible case, preferential concentration occurs in regions of negative divergence of the field. In the incompressible case, it occurs in regions of simultaneously high strain and negative field skewness.

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I. INTRODUCTION

One of the most striking characteristics of inertial particle transport by random velocity fields is clustering. This phenomenon occurs, but is not confined to, in turbulent flows [1]; clustering phenomena, in fact, were initially predicted to occur with particles pushed by a one-dimensional (1D) random force field [2]. The interesting point is that an initial spatially homogeneous distribution of inertial particles will develop clumps and voids, even if the flow is incompressible. Both experimental evidence [1] and numerical simulations [3,4] confirm this effect. The spatial inhomogeneity of the random-field statistics may contribute, but is not crucial, to the process. It is to be mentioned that clustering phenomena are thought to be important both in industrial flows [5], in the atmosphere (the problem of rain formation) [6,7], and in the oceans (the problem of plankton dynamics, especially as regard to blooming) [8,9].

Over the years, a substantial theoretical effort has been directed to the analysis of clustering by random flows [10–17]. As first noticed in [2], clustering appears to be associated with a weak inertia regime, in which the separation of the trajectories of the particles and the fluid elements they cross in their motion is small on the scale of the trajectory evolution. This corresponds to a situation in which the relaxation time of the particle velocity is shorter than the characteristic time of the random-field fluctuations. In the absence of molecular diffusion, the resulting clusters are of singular nature, concentrated on a set of zero measure [2,18].

An approach that has been fruitful in the study of passive tracer transport is that of considering uncorrelated in time random velocity fields, the so-called Kraichnan model [19]. The role of characteristic time of the random field is played in this case by the diffusion time of a tracer (or, depending on the problem, pairs of tracers) across a correlation length of the field. Weak and strong inertia will then refer to fast or slow relaxation with respect to this characteristic time. The Kraichnan model approach, in the case of inertial particle transport, has allowed the derivation of analytical expressions for the particle concentration correlations, both in the weak [17,20] and large [21] inertia limits. As recognized in [17], the weak inertia limit, in a Kraichnan model approach, corresponds to a regime of adiabatic variation for the particle separation. In this framework, the particle concentration dynamics can be cast in the form of a problem of fast variable elimination, in which the fast variables are the particle velocities.

A mechanism that has been proposed for cluster formation in turbulence is the centrifugal force induced preferential concentration of heavy (light) particles in the strain (vorticity) regions of the fluid [22]. This preferential concentration effect was later confirmed in numerical simulations [3,23]. Now, clustering turns out to occur also in one dimension and in the Kraichnan model just discussed in situations, therefore, where it is not clear what meaning should be given to preferential concentration. In particular, the concepts of strain and vorticity do not exist in one dimension. This casts some doubts on whether preferential concentration (or some generalized version of it) is an essential ingredient for inertial particle clustering in random flows.

In general, preferential concentration could be defined as the fact that averages of fluid (random flow) quantities, obtained from sampling along inertial particles trajectories, do not coincide with what would be obtained from spatial (or temporal) averages. In other words, it could be interpreted as a nonergodicity property of the process of random-field sampling by the particle flow.

Given the situation, a first question that would be interesting to ask is whether the clustering of inertial particles in random flows is always the result of the nonergodic sampling, by the particles, of some relevant field quantity. In the presence of finite correlation time and incompressibility, physical considerations allowed us to identify from the start, strain and vorticity as the relevant quantities. In the general case, such an operation may be not as easy, and an interesting question is, therefore, whether the correct relevant quantities could be identified directly from the equations of motion.

This paper will try to answer these two questions. The analysis will show that the answer is part of the procedure of fast variable elimination required to determine the dynamics of the particle concentration field. It is actually the way in which the fast variable elimination procedure handles the presence of memory in the original process.

This paper is organized as follows. In Sec. II, the equations for the transport of a pair of inertial particle in a random field are derived. In Sec. III, evolution equations for the particle separation distribution are derived and applied to the determination of the clustering strength. In Sec. IV the issue of preferential concentration and its contribution to clustering is discussed. Section V is devoted to conclusions. Technical details are left in the appendixes.

II. MODEL EQUATIONS

The motion of an inertial particle in a random velocity field $\mathbf{u}(\mathbf{x},t)$ can be modeled by the Stokes equation

$$\dot{\mathbf{v}} = \tau_S^{-1}[\mathbf{u}(\mathbf{x},t) - \mathbf{v}], \quad \dot{\mathbf{x}} = \mathbf{v}. \tag{1}$$

In fluid mechanics, this model would describe the dynamics of a particle that is sufficiently small, of sufficiently high density [24], and in flow conditions in which the effect of gravity can be neglected. The relaxation time τ_S, called the Stokes time, in the case of a spherical particle is given by $\tau_S=(2/9)(a^2\lambda/\nu_0)$, where a is the particle radius, λ is the ratio of the particle and fluid densities, and ν_0 is the fluid kinematic viscosity.

Consider a smooth D-dimensional Gaussian velocity field $\mathbf{u}(\mathbf{x},t)$, with stationary and spatially uniform and isotropic statistics. For an incompressible random field, the structure function $\langle\hat{u}_\alpha(\mathbf{r},t)\hat{u}_\beta(\mathbf{r},0)\rangle$, $\hat{\mathbf{u}}(\mathbf{r},t)\equiv\mathbf{u}(\mathbf{x}+\mathbf{r},t)-\mathbf{u}(\mathbf{x},t)$, can be written in the form $\langle\hat{u}_\alpha(\mathbf{r},t)\hat{u}_\beta(\mathbf{r},0)\rangle=F(t)\hat{g}_{\alpha\beta}(\mathbf{r})$, where, for $r\ll r_{\mathbf{v}}$,

$$\hat{g}_{\alpha\beta}(\mathbf{r}) = \frac{\sigma_u^2\tau_E}{r_{\mathbf{v}}^2}[(D+1)r^2\delta_{\alpha\beta} - 2r_\alpha r_\beta], \tag{2}$$

and $\int_0^\infty|F(t)|dt=1$, with σ_u, $r_{\mathbf{v}}$, and τ_E being, respectively, the characteristic velocity, length, and time scales of the field. Following [16], the following dimensionless parameters are introduced:

$$S = \tau_S/\tau_E, \quad K = \sigma_u\tau_E/r_{\mathbf{v}}, \tag{3}$$

which are called, respectively, Stokes and Kubo numbers. Units are chosen such that $\sigma_u=\tau_S=1$; in this way,

$$\tau_E = S^{-1}, \quad r_{\mathbf{v}} = (KS)^{-1}. \tag{4}$$

The Kubo number describes the intrinsic long- or short-correlated in time nature of the field, with $K\to 0$ $(S\to\infty)$ corresponding to an uncorrelated regime: $F(t)\to 2\delta(t)$; $K\to\infty$ $(S\to 0)$ corresponds in turn to frozen field conditions. For real turbulence, $K\sim 1$ and S parametrizes the strength of inertia.

The $K\to 0$ regime, corresponding to a Kraichnan model for inertial particles, allows us to neglect the particle displacement in a correlation time τ_E. Clustering is a phenomenon associated with singular behavior, at small values of the argument, of the particle separation probability density function (PDF) $\rho(\mathbf{r},t)$. A Kraichnan model approach allows us to describe the two-particle dynamics relevant for clustering, in terms of a single equation,

$$\dot{\boldsymbol{\nu}} = \hat{\mathbf{u}}(\mathbf{r},t) - \boldsymbol{\nu}, \quad \dot{\mathbf{r}} = \boldsymbol{\nu}, \tag{5}$$

as the evolution of $\hat{\mathbf{u}}(\mathbf{r},t)$ depends only on $\mathbf{r}$, and not separately on the coordinates of the two particles. [For finite τ_E, this would not be true, as $\hat{\mathbf{u}}(r(t),t)$ would depend, on scale τ_E, on the separate evolution of the two-particle coordinates, described by Eq. (1)]. In a Kraichnan model approach, the separation PDF $\rho(\mathbf{r},t)$ will obey the Fokker-Planck equation associated with Eq. (5) (more precisely, its restriction to the variable $\mathbf{r}$).

As recognized in [16], the two-particle dynamics becomes dependent, in an uncorrelated in time regime, on the single parameter

$$\epsilon = \frac{\sigma_u^2\tau_E}{r_{\mathbf{v}}^2} = K^2S, \tag{6}$$

which is the amplitude factor in front of Eq. (2). This parameter plays the role of generalized Stokes number for a Kraichnan model. Writing $\epsilon=\tau_S/\tilde{\tau}_E$, one notices in fact that $\tilde{\tau}_E=r_{\mathbf{v}}^2/(\sigma_u^2\tau_E)$ is the time for a tracer [a point moving with velocity $\mathbf{u}(\mathbf{x}(t),t)$] to diffuse across $r_{\mathbf{v}}$ that plays the role of characteristic time scale for the uncorrelated in time random flow.

Small ϵ corresponds to a weak inertia regime, and Eqs. (2), (5), and (6) provide, for $\tau_E\to 0$, a Kraichnan model for the clustering of weakly inertial particles (see [16,17,20] and references therein). To see that small ϵ corresponds to weak inertia, it suffices to verify that r changes little on a time τ_S (the correlation time for $\boldsymbol{\nu}$): $\Delta r(\tau_S)\ll r$. This is verified *a posteriori* solving Eqs. (2) and (5) for fixed $\mathbf{r}$, $r\ll r_{\mathbf{v}}$. This gives $\langle\nu^2|r\rangle\sim\epsilon r^2$ that coincides—in the dimensionless units of Eq. (4)—with the square displacement in a time τ_S. Thus, $\Delta r(\tau_S)/r\sim\epsilon^{-1/2}\ll 1$, as expected.

This is an adiabatic regime for $\mathbf{r}$ that has allowed the authors in [17] to use a fast variable elimination technique to derive a version of the Fokker-Planck equation associated with Eq. (5), restricted to $\mathbf{r}$. In the following sections, a similar approach will be utilized to establish the connection between clustering and preferential concentration effects.

III. CLUSTERING

The fast variable elimination procedure in a stochastic problem, like the one described by Eq. (5) in the $\epsilon\ll 1$ regime, can be carried on substantially in two ways [25]: working at the Fokker-Planck equation level, by means of the so-called projection operator techniques [26,27], or averaging away the fast variables already at the level of the stochastic differential equation. The procedure followed in [17] was of the projection operator type, along the lines of the approach derived in [28], in the context of stochastic climate modeling. This approach is not appropriate here: dealing with preferential concentration effects will require the evaluation of conditional averages such as $\langle f_n[\hat{\mathbf{u}}]|\mathbf{r}(t)=\bar{\mathbf{r}}\rangle$, with $f_n[\hat{\mathbf{u}}]$ products of the fields and their derivatives, about which the Fokker-Planck equation associated with Eq. (5) provides no information.

The approach that is going to be followed here is the one described in [29,30], based on the use of the so-called stochastic Liouville equation and of functional derivation techniques [31,32]. A similar approach was followed in [21] to analyze the large inertia limit of particle transport.

The evolution equation for the PDF $\rho(\bar{\mathbf{r}},t)$, associated with Eq. (5), can be written in the form $\partial_t\rho(\bar{\mathbf{r}},t)+\bar{\partial}_\alpha J_\alpha(\bar{\mathbf{r}},t)=0$ (where $\bar{\partial}_\alpha\equiv\partial/\partial\bar{r}_\alpha$), with the probability current J_α given by

$$J_\alpha(\bar{\mathbf{r}},t)=\int_{-\infty}^{t}d\tau\, e^{\tau-t}\langle\hat{u}_\alpha(\mathbf{r}(\tau),\tau)\delta(\mathbf{r}(t)-\bar{\mathbf{r}})\rangle. \tag{7}$$

This evolution equation is basically the average over all the realizations of $\hat{\mathbf{u}}$ of the Liouville equation in the configuration space of Eq. (5): $\partial_t\tilde{\rho}(\bar{\mathbf{r}},t)+\bar{\partial}_\alpha[\nu_\alpha(t)\tilde{\rho}(\bar{\mathbf{r}},t)]=0$, with $\tilde{\rho}(\bar{\mathbf{r}},t)=\delta(\mathbf{r}(t)-\bar{\mathbf{r}})$. The integral $\int_{-\infty}^{t}d\tau\, e^{\tau-t}\hat{\mathbf{u}}(\mathbf{r}(\tau),\tau)$ in Eq. (7), in fact, is just the solution for $\boldsymbol{\nu}(t)$ of Eq. (5), and $\mathbf{J}\equiv\langle\boldsymbol{\nu}(t)|\mathbf{r}(t)=\bar{\mathbf{r}}\rangle\rho(\bar{\mathbf{r}},t)$, with $\rho(\bar{\mathbf{r}},t)=\langle\tilde{\rho}(\bar{\mathbf{r}},t)\rangle$.

The correlation between the Dirac delta and the random field in Eq. (7) is calculated using the functional integration by part formula [31,32] (see also [33]). Indicating by $\delta/\delta\hat{\mathbf{u}}(\mathbf{x},t)$ the operation of functional differentiation, this corresponds to making in Eq. (7) the substitution

$$\hat{u}_\alpha(\mathbf{z},t)\to 2\int d^Dr'\ \hat{g}_{\alpha\beta}(\mathbf{z},\mathbf{z}')\frac{\delta}{\delta\hat{u}_\beta(\mathbf{z}',t)}, \tag{8}$$

where $2\hat{g}_{\alpha\beta}(\mathbf{z},\mathbf{z}')\delta(t-t')=\langle\hat{u}_\alpha(\mathbf{z},t)\hat{u}_\beta(\mathbf{z}',t')\rangle$ and of course $\hat{g}_{\alpha\beta}(\mathbf{z},\mathbf{z})\equiv\hat{g}_{\alpha\beta}(\mathbf{z})$ [34]. In order to use Eq. (8), however, it is first necessary to write in Eq. (7) $\hat{u}_\alpha(\mathbf{r}(\tau),\tau)=\int d^Dz\ \hat{u}_\alpha(\mathbf{z},\tau)\delta(\mathbf{r}(\tau)-\mathbf{z})$; the functional derivative $\delta/\delta\hat{u}_\beta(\mathbf{z}',\tau)$ will then act on a product $\delta(\mathbf{r}(\tau)-\mathbf{z})\delta(\mathbf{r}(t)-\bar{\mathbf{r}})$. This requires the determination of expressions like

$$R_{\gamma\beta}(t;\mathbf{z}',\tau)=\frac{\delta r_\gamma(t)}{\delta\hat{u}_\beta(\mathbf{z}',\tau)}. \tag{9}$$

Writing $\delta/\delta\hat{u}_\beta(\mathbf{z}',\tau)\delta(\mathbf{r}(0)-\bar{\mathbf{r}})=-R_{\gamma\beta}(t;\mathbf{z}',\tau)\bar{\partial}_\gamma\delta(\mathbf{r}(t)-\bar{\mathbf{r}})$, it is clear that the role of the response function $R_{\gamma\beta}(t;\mathbf{z}',\tau)$ in Eq. (7) is precisely to account for the correlation between the random field and the condition on the separation at time t.

Following the approach in [29], outlined in Appendix A, the response function can be calculated as an expansion in powers of $\hat{\mathbf{u}}$ [i.e., from Eqs. (2) and (6), basically in powers of $\epsilon^{1/2}$]: $R_{\gamma\beta}(t;\mathbf{z},\tau)=\delta(\mathbf{r}(\tau)-\mathbf{z})[\hat{R}^{(0)}_{\gamma\beta}+\hat{R}^{(1)}_{\gamma\beta}+\cdots]$. The first two terms in the expansion read [see Eqs. (A3) and (A4)]

$$\hat{R}^{(0)}_{\gamma\beta}=\psi(\tau-t)\delta_{\gamma\beta},\qquad \psi(t)=\theta(-t)[1-e^{t}], \tag{10}$$

$$\hat{R}^{(1)}_{\gamma\beta}=\int_{\tau}^{t}d\tau'\ \psi(\tau-\tau')\psi(\tau'-t)\frac{\partial\hat{u}_\gamma(\mathbf{r}(\tau'),\tau')}{\partial r_\beta(\tau')}, \tag{11}$$

and $\theta(t)$ is the Heaviside step function [$\theta(t)=1$ for $t>0$; $\theta(t)=0$, otherwise].

From Eqs. (10), (11), and (A4), it appears that $R_{\gamma\beta}(\tau;\mathbf{z}',\tau)=0$, so that the functional derivative on $\delta(\mathbf{r}(\tau)-\mathbf{z})$, arising in Eq. (7) from $\hat{u}_\alpha(\mathbf{r}(\tau),\tau)=\int d^Dz\ \hat{u}_\alpha(\mathbf{z},\tau)\delta(\mathbf{r}(\tau)-\mathbf{z})$, does not contribute. The analysis in Appendix A shows that the lowest-order inertia contribution to J_α occurs at $O(\epsilon^2)$, corresponding to taking into account only $R^{(1)}_{\gamma\beta}(t;\mathbf{z}',\tau)$ in the expansion for $R_{\gamma\beta}(t;\mathbf{z}',\tau)$. Carrying out the necessary functional differentiations in Eq. (7) and using Eqs. (10) and (11) gives then the result, for $t=0$,

$$J_\alpha\simeq-2\bar{\partial}_\gamma\int_{-\infty}^{0}d\tau\, e^{\tau}\Bigg\langle\hat{g}_{\alpha\beta}(\mathbf{r}(\tau))\delta(\mathbf{r}(0)-\bar{\mathbf{r}})\Bigg[\delta_{\gamma\beta}\psi(\tau)+\int_{\tau}^{0}d\tau'\ \psi(\tau-\tau')\psi(\tau')\frac{\partial\hat{u}_\gamma(\mathbf{r}(\tau'),\tau')}{\partial r_\beta(\tau')}\Bigg]\Bigg\rangle. \tag{12}$$

The lowest-order contribution to the current is, exploiting incompressibility, $\bar{\partial}_\gamma\hat{g}_{\alpha\gamma}(\bar{\mathbf{r}})=0$,

$$J^{(2)}_\alpha=-g_{\alpha\gamma}(\bar{\mathbf{r}})\bar{\partial}_\gamma\rho(\bar{\mathbf{r}},t), \tag{13}$$

corresponding, in Eq. (12), to putting $\hat{g}_{\alpha\beta}(\mathbf{r}(\tau))\simeq\hat{g}_{\alpha\beta}(\mathbf{r}(0))$ and to neglect the term $\propto\partial_\beta\hat{u}_\gamma$ in the integral. Equation (13) describes pure tracer transport.

The next order is $J^{(4)}$ that receives two contributions: the one $\propto\partial_\beta\hat{u}_\gamma$, originating from $\hat{R}^{(1)}$, and the one $\propto\hat{g}_{\alpha\beta}(\mathbf{r}(\tau))-\hat{g}_{\alpha\beta}(\mathbf{r}(0))$. The first contribution is calculated by applying Eq. (8) to the factor $\partial\hat{u}_\gamma(\mathbf{r}(\tau'),\tau')/\partial r_\beta(\tau')$ in Eq. (12). The second contribution is calculated by writing $\hat{g}_{\alpha\beta}(\mathbf{r}(\tau))-\hat{g}_{\alpha\beta}(\mathbf{r}(0))$ as a function of $\Delta\mathbf{r}(\tau)=\mathbf{r}(\tau)-\mathbf{r}(0)$ and using the expression, which descends from Eq. (5),

$$\Delta\mathbf{r}(\tau)=\int_{-\infty}^{0}d\tau'\ \Phi(\tau',\tau)\hat{\mathbf{u}}(\mathbf{r}(\tau'),\tau'), \tag{14}$$

where

$$\Phi(\tau',\tau)=\psi(\tau'-\tau)-\psi(\tau'). \tag{15}$$

Evaluating to $O(\epsilon^2)$ the two contributions and exploiting incompressibility leads to the result (see Appendix B)

$$J^{(4)}_\alpha=-\tfrac{1}{2}\rho(\bar{\mathbf{r}})\bar{\partial}_\beta\bar{\partial}_\eta\hat{g}_{\alpha\gamma}(\bar{\mathbf{r}})\bar{\partial}_\gamma\hat{g}_{\beta\eta}(\bar{\mathbf{r}}) \tag{16}$$

plus other terms containing spatial derivatives of ρ. The stationary solution $\rho(\bar{\mathbf{r}})$ is obtained requiring that the current be divergence free: $\partial_\alpha J_\alpha=0$. The lowest-order solution, from Eq. (13), is spatially uniform, and the derivatives acting on ρ do not contribute to Eq. (16), at the order considered. The stationary solution will obey, therefore, the equation

$$\hat{g}_{\alpha\gamma}(\bar{\mathbf{r}})\bar{\partial}_\alpha\bar{\partial}_\gamma\rho(\bar{\mathbf{r}})+\frac{\rho(\bar{\mathbf{r}})}{2}\bar{\partial}_\beta\bar{\partial}_\eta\hat{g}_{\alpha\gamma}(\bar{\mathbf{r}})\bar{\partial}_\alpha\bar{\partial}_\gamma\hat{g}_{\beta\eta}(\bar{\mathbf{r}})=0, \tag{17}$$

which is in agreement with [17], once a wrong sign in their Eq. (40) is corrected [35]. As discussed in [17,20], the solution of Eq. (17), with the expression for $\hat{g}_{\alpha\beta}(\mathbf{r})$ provided in Eq. (2), is a power law at $r\to 0$: $\rho(\mathbf{r})\propto r^{-c\epsilon}$, with $c=2(D+1)(D+2)$, corresponding to the correlation dimension for the particle distribution: $D_2=D-2(D+1)(D+2)\epsilon$.

IV. PREFERENTIAL CONCENTRATION

In the previous section, clustering was obtained by solving an evolution equation for the separation PDF $\rho(\bar{\mathbf{r}},t)$: $\partial_t\rho+\bar{\partial}_\alpha J_\alpha=0$, in which the probability current divergence $\bar{\partial}_\alpha J_\alpha$ was expressed, through Eq. (7), in terms of a conditional average $\bar{\partial}_\alpha\langle\hat{u}_\alpha(\mathbf{r}(\tau),\tau)\delta(\mathbf{r}(0)-\bar{\mathbf{r}})\rangle$. Since, in the absence of conditioning, this average would be zero, it is clear that nonergodic sampling of the random field along the particle trajectories is a necessary condition for a nonzero current, and hence for clustering.

In order to clearly identify the quantities undergoing preferential concentration, it is necessary to pass the divergence operator in $\bar{\partial}_\alpha\langle\hat{u}_\alpha(\mathbf{r}(\tau),\tau)\delta(\mathbf{r}(0)-\bar{\mathbf{r}})\rangle$, inside the average, so that the expression can be written in the form $\langle f[\hat{\mathbf{u}}]\delta(\mathbf{r}(0)-\bar{\mathbf{r}})\rangle$. From $f[\hat{\mathbf{u}}]$, one can then extract the contributions from specific monomials $f_n[\hat{\mathbf{u}}]$ in the fields and their derivatives.

In the present situation, a good strategy to identify these monomials is to expand the field $\hat{u}_\alpha(\mathbf{r}(\tau),\tau)$ around the final point $\mathbf{r}(0)$,

$$\hat{\mathbf{u}}(\mathbf{r}(\tau),\tau)=\Big\{1+\Delta r_\beta^{(1)}(\tau)\partial_\beta+\Delta r^{(2)}(\tau)\partial_\beta + \tfrac{1}{2}\Delta r_\beta^{(1)}(\tau)\Delta r_\gamma^{(1)}(\tau)\partial_\beta\partial_\gamma+\cdots\Big\}\hat{\mathbf{u}}(\mathbf{r}(0),\tau), \tag{18}$$

where

$$\Delta\mathbf{r}^{(1)}(\tau)=\hat{\Phi}\hat{\mathbf{u}}(\mathbf{r}(0),\tau), \tag{19}$$

$$\Delta\mathbf{r}^{(2)}(\tau)=\hat{\Phi}\Delta r_\beta^{(1)}\partial_\beta\hat{\mathbf{u}}(\mathbf{r}(0),\tau), \tag{20}$$

and so on to higher orders, with $\hat{\Phi}g(\tau)\equiv\int_{-\infty}^{0}d\tau'\ \Phi(\tau',\tau)g(\tau')$ and $\Phi(\tau,\tau')$ given in Eq. (15). In the weak inertia regime considered, $\epsilon\ll 1$, this is appropriate in the interval $-\tau\lesssim 1$, in which the time integral in Eq. (7) is concentrated.

Substituting Eqs. (18)–(20) into Eq. (7), the current divergence can be written as a sum of averages, involving increasing powers of $\hat{\mathbf{u}}$. The first contributions are

$$f_1[\hat{\mathbf{u}}]=\bar{\partial}_\alpha\hat{u}_\alpha(\bar{\mathbf{r}},\tau), \tag{21}$$

$$f_2[\hat{\mathbf{u}}]=\bar{\partial}_\alpha\bar{\partial}_\beta[\hat{u}_\alpha(\hat{\mathbf{r}},\tau)\hat{u}_\beta(\hat{\mathbf{r}},\tau')], \tag{22}$$

$$f_3[\hat{\mathbf{u}}]=\bar{\partial}_\alpha\bar{\partial}_\beta\bar{\partial}_\gamma[\hat{u}_\alpha(\bar{\mathbf{r}},\tau)\hat{u}_\beta(\bar{\mathbf{r}},\tau')\hat{u}_\gamma(\bar{\mathbf{r}},\tau'')]. \tag{23}$$

Other contributions involve gradients of the Dirac delta in Eq. (7) that would lead in the end to gradients of $\rho(\bar{\mathbf{r}},t)$. The terms that would lead to clustering even starting from a spatially homogeneous particle distribution, however, are those in Eqs. (21)–(23).

The contribution from Eq. (21) is evaluated following the same procedure leading from Eq. (7) to Eq. (12). Stopping to lowest order in ϵ and neglecting terms involving gradients of ρ leads to the result (see Appendix C)

$$\langle\bar{\partial}_\alpha\hat{u}_\alpha(\bar{\mathbf{r}},\tau)|\mathbf{r}(0)=\bar{\mathbf{r}}\rangle=-\tfrac{1}{2}\psi(\tau)\bar{\partial}_\alpha\bar{\partial}_\gamma\hat{g}_{\alpha\gamma}(\bar{\mathbf{r}}) \tag{24}$$

that will vanish if $\mathbf{u}$ is incompressible. This means that, in the case of a compressible random field, particularly in $D=1$, clustering will be the result of preferential concentration in regions of negative divergence of $\mathbf{u}$. More precisely, particle pairs that at time $t=0$ have separation $\bar{\mathbf{r}}$, in the past were more likely to be in regions of negative $\nabla\cdot\mathbf{u}$.

It is to be noticed that Eq. (18) does not correspond to an expansion of J_α in powers of $\epsilon^{1/2}$. The fields $\hat{\mathbf{u}}$ in Eq. (18), upon substitution in Eq. (7), contribute $O(\epsilon^{1/2})$ in contraction with another field $\hat{\mathbf{u}}$, but contribute $O(\epsilon)$ in contraction with $\delta(\mathbf{r}(0)-\bar{\mathbf{r}})$. A consequence of this is that Eq. (24) does not provide the whole $O(\epsilon)$ part of $\bar{\partial}_\alpha J_\alpha$. In the compressible case, another $O(\epsilon)$ contribution to $\bar{\partial}_\alpha J_\alpha$ is provided, for instance, by $\bar{\partial}_\alpha\bar{\partial}_\beta\hat{g}_{\alpha\beta}(\bar{\mathbf{r}})$ that comes from the part of $\langle\bar{\partial}_\alpha\bar{\partial}_\beta[\hat{u}_\alpha(\bar{\mathbf{r}},\tau)\hat{u}_\beta(\bar{\mathbf{r}},\tau')]|\mathbf{r}(0)=\bar{\mathbf{r}}\rangle$ that is uncorrelated with $\delta(\mathbf{r}(0)-\bar{\mathbf{r}})$.

In the incompressible case, the contribution from f_1 to the current divergence vanishes, and it is necessary to consider f_n, with $n>1$, in particular, the term in Eq. (22). Now, the strain and the vorticity variance of a velocity field $\mathbf{u}$ are defined from

$$|\mathbf{S}|^2=(1/2)\partial_\alpha u_\beta(\partial_\alpha u_\beta+\partial_\beta u_\alpha),$$

$$|\boldsymbol{\omega}|^2=\partial_\alpha u_\beta(\partial_\alpha u_\beta-\partial_\beta u_\alpha), \tag{25}$$

and $\partial_\alpha u_\beta\partial_\beta u_\alpha=|\mathbf{S}|^2-|\boldsymbol{\omega}|^2/2$, whose average can be shown to be zero if $\mathbf{u}$ is incompressible. Thus, $f_2[\hat{\mathbf{u}}]$ is connected with the difference $|\mathbf{S}|^2-|\boldsymbol{\omega}|^2/2$ for $\mathbf{u}$.

Since, in the case of an incompressible field, $\langle|\mathbf{S}|^2\rangle=\langle|\boldsymbol{\omega}|^2/2\rangle$, $\langle f_2[\hat{\mathbf{u}}]|\mathbf{r}(0)=\bar{\mathbf{r}}\rangle$ will receive its first nonzero contribution by contraction of the two fields $\hat{u}_\alpha$ and $\hat{u}_\beta$ in f_2 with $\delta(\mathbf{r}(0)-\bar{\mathbf{r}})$ [see Eq. (22)]. The calculation carried on in Appendix C gives the result, for $\tau>\tau'$,

$$\langle\bar{\partial}_\alpha\hat{u}_\beta(\bar{\mathbf{r}},\tau')\bar{\partial}_\beta\hat{u}_\alpha(\bar{\mathbf{r}},\tau)|\mathbf{r}(0)=\bar{\mathbf{r}}\rangle = \psi(\tau)[3\psi(\tau')-\psi(\tau'-\tau)]\bar{\partial}_\alpha\bar{\partial}_\beta\hat{g}_{\gamma\eta}(\bar{\mathbf{r}})\bar{\partial}_\gamma\bar{\partial}_\eta\hat{g}_{\alpha\beta}(\bar{\mathbf{r}}) \tag{26}$$

plus terms involving gradients of $\rho(\bar{\mathbf{r}},t)$. The quantity to the right-hand side (RHS) of Eq. (26) is positive for $\tau'<\tau<0$. Thus, preferential concentration in regions of high strain, in the wider sense given here to the term, is a property that is maintained in an incompressible Kraichnan model for inertial particle transport.

Again, Eq. (26) does not account for the whole of the current divergence to $O(\epsilon^2)$ in the incompressible case. One has to consider also the skewnesslike term in Eq. (23). This time, it is necessary to consider the contraction of only one of the fields with $\delta(\mathbf{r}(0)-\bar{\mathbf{r}})$. A calculation completely analogous to that of Eq. (24) gives then the result

$$\langle\bar{\partial}_\alpha\bar{\partial}_\beta\bar{\partial}_\gamma[\hat{u}_\alpha(\bar{\mathbf{r}},\tau)\hat{u}_\beta(\bar{\mathbf{r}},\tau')\hat{u}_\gamma(\bar{\mathbf{r}},\tau'')]|\mathbf{r}(0)=\bar{\mathbf{r}}\rangle = -\bar{\partial}_\alpha\bar{\partial}_\beta\hat{g}_{\gamma\eta}(\bar{\mathbf{r}})\bar{\partial}_\gamma\bar{\partial}_\eta\hat{g}_{\alpha\beta}(\bar{\mathbf{r}}) \times[\psi(\tau)\delta(\tau'-\tau'')+\text{permutations}] \tag{27}$$

plus, again, terms involving gradients of $\rho(\bar{\mathbf{r}},t)$. Thus, clustering receives, in the incompressible case, contribution from preferential concentration in regions of simultaneously high strain and negative skewness $\bar{\partial}_\alpha\bar{\partial}_\beta\bar{\partial}_\gamma[\hat{u}_\alpha(\bar{\mathbf{r}},\tau)\hat{u}_\beta(\bar{\mathbf{r}},\tau')\hat{u}_\gamma(\bar{\mathbf{r}},\tau'')]$.

The terms in Eqs. (26) and (27) are all the preferential concentration contributions that arise, at $O(\epsilon^2)$, in the incompressible case. A term f_4 of fourth order in $\hat{\mathbf{u}}$, not indicated in Eq. (18), may contribute to $\langle f_4[\hat{\mathbf{u}}]|\mathbf{r}(0)=\bar{\mathbf{r}}\rangle$; at $O(\epsilon^2)$, this occurs through the unconditional average $\langle f_4[\hat{\mathbf{u}}]\rangle$. However, Gaussianity of $\mathbf{u}$ reduces this contribution to products $\langle f_2[\hat{\mathbf{u}}]\rangle\langle f_2[\hat{\mathbf{u}}]\rangle$, which vanish in the case of an incompressible random field.

It is to be noticed that the RHSs of Eqs. (24) and (23) are independent of the argument $\bar{\mathbf{r}}$. This means that the averages in those equations are conditioned to the presence of a pair of particles at unspecified separation. In other words, the two-particle conditional averages in Eqs. (24)–(27) are equivalent to averages conditioned to a single particle in $\mathbf{x}+\bar{\mathbf{r}}$.

V. CONCLUSION

The analysis in this paper shows that some generalized form of preferential concentration continues to be an ingredient for clustering, also in situations where its presence is not expected, such as transport by uncorrelated in time or 1D random velocity fields. The original statement that the random field at the particle location has on the average some property (e.g., higher strain) is replaced, in the uncorrelated case, with the one on properties of the random field, still at that location, but at times previous to the arrival of the particle. The current field configuration, instead, is uncorrelated with the current particle positions. In other words, if the field is uncorrelated in time, there will be no preferential concentration, in the standard meaning of the term, rather, a generalized version involving field and particle configurations at different times.

The weak inertia considered has allowed us to identify the quantities involved in this form of generalized preferential concentration, through an expansion in the effective Stokes number ϵ [see Eqs. (3), (4), and (6)]. In the two regimes of compressible and incompressible flows, the relevant field properties are related to the field divergence at the particle position, in the first case, and to the difference between the square strain and vorticity in the second. An additional relevant quantity in the incompressible case, not previously considered in the literature (to the author's knowledge), is the skewness in Eq. (27).

The standard interpretation of preferential concentration is recovered considering a local time average of the flow: $\bar{\mathbf{u}}(\bar{\mathbf{r}},t)=\Delta t^{-1}\int_{-\Delta t/2}^{\Delta t/2}\hat{\mathbf{u}}(\bar{\mathbf{r}},t+\tau)d\tau$. Basically, this gives to the random field a finite correlation time Δt. From Eqs. (21)–(23), it is then possible to define time averaged quantities $\bar{\partial}_\alpha\bar{u}_\alpha(\bar{\mathbf{r}},\tau)$, $\bar{\partial}_\alpha\bar{\partial}_\beta[\bar{u}_\alpha(\bar{\mathbf{r}},\tau)\bar{u}_\beta(\bar{\mathbf{r}},\tau)]$, and $\bar{\partial}_\alpha\bar{\partial}_\beta\bar{\partial}_\gamma[\bar{u}_\alpha(\bar{\mathbf{r}},\tau)\bar{u}_\beta(\bar{\mathbf{r}},\tau)\bar{u}_\gamma(\bar{\mathbf{r}},\tau)]$, corresponding to divergence, difference between strain and vorticity squared [see Eq. (25)], and skewness of the field $\bar{\mathbf{u}}$. From Eqs. (24)–(27), it is easy to see that inertial particles, with respect to these quantities, undergo preferential concentration in the usual sense of the word.

The analogy between particle transports by a Kraichnan model and by a random velocity field with finite correlation time, with the parameter ϵ in the first case set equal to S in the second, allows predictions of the preferential concentration strength in the finite correlation time case. Indicating by $\langle\cdot\rangle_p$ the average at a particle position, $(r_\mathbf{v}/\sigma_u)\langle\nabla\cdot\mathbf{u}\rangle_p=O(S^{1/2})$, in the compressible case, and $(r_\mathbf{v}/\sigma_u)^2\langle(|\mathbf{S}|^2-|\boldsymbol{\omega}|^2/2)\rangle_p=O(S)$, $(r_\mathbf{v}/\sigma_u)^3\langle\partial_\alpha\partial_\beta\partial_\gamma[u_\alpha u_\beta u_\gamma]\rangle_p=O(S^{1/2})$ in the incompressible one. Preferential concentration should occur in regions of negative field divergence in the compressible case, and high strain and negative skewness in the incompressible one.

The techniques utilized in this paper are not specific to inertial particles and are in fact rather general. For instance, conclusions analogous to the ones on inertial particles could be drawn, in the case of fluid tracers in compressible flows, both regarding clustering and preferential concentration. (In fact, it would be interesting to understand how much of particle clustering in compressible flows is associated with inertia and how much is due to accumulation on fluid shocks.)

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APPENDIX A: RESPONSE FUNCTION DETERMINATION

Following [29], an equation for the response function $R_{\gamma\beta}(t;\mathbf{z},\tau)=\delta r_\gamma(t)/\delta\hat{u}_\beta(\mathbf{z},\tau)$ can be obtained by writing Eq. (5) in the form

$$\ddot{r}_\gamma(t)+\dot{r}_\gamma(t)=\hat{u}_\gamma(\mathbf{r}(t),t)$$

and taking the functional derivative with respect to $\hat{u}_\beta(\mathbf{z},\tau)$. The result is, for $t>\tau$,

$$\ddot{R}_{\gamma\beta}(t;\mathbf{z},\tau)+\dot{R}_{\gamma\beta}(t;\mathbf{z},\tau)=\frac{\partial\hat{u}_\gamma(\mathbf{r}(t),t)}{\partial r_\eta(t)}R_{\eta\beta}(t;\mathbf{z},\tau). \quad \text{(A1)}$$

From Eq. (5), one can write $\mathbf{r}(t)=\text{const}+\int_{-\infty}^{t}d\tau\,\psi(\tau-t)\hat{\mathbf{u}}(\mathbf{r}(\tau),\tau)$ and $\boldsymbol{\nu}(t)=\int_{-\infty}^{t}d\tau\,e^{\tau-t}\hat{\mathbf{u}}(\mathbf{r}(\tau),\tau)$, with $\psi(t)$ given in Eq. (10). These expressions lead to the initial conditions

$$R_{\gamma\beta}(\tau;\mathbf{z},\tau)=0,\quad \dot{R}_{\gamma\beta}(\tau;\mathbf{z},\tau)=\delta_{\gamma\beta}\delta(\mathbf{r}(\tau)-\mathbf{z}). \quad \text{(A2)}$$

Equation (A1) with the initial conditions in Eq. (A2) can be solved perturbatively in $\hat{\mathbf{u}}$ (i.e., basically in $\epsilon^{1/2}$): $R_{\gamma\beta}(t;\mathbf{z},\tau)=\delta(\mathbf{r}(\tau)-\mathbf{z})[\hat{R}^{(0)}_{\gamma\beta}+\hat{R}^{(1)}_{\gamma\beta}+\cdots]$, with the result

$$\hat{R}^{(0)}_{\gamma\beta}=\psi(\tau-t)\,\delta_{\gamma\beta}, \quad \text{(A3)}$$

$$\hat{R}^{(n+1)}_{\gamma\beta}=\int_\tau^t d\tau'\;\psi(\tau'-t)\frac{\partial\hat{u}_\gamma(\mathbf{r}(\tau'),\tau')}{\partial r_\eta(\tau')}\hat{R}^{(n)}_{\eta\beta}. \quad \text{(A4)}$$

In particular, the second-order term reads

$$\hat{R}^{(2)}_{\gamma\beta}=\int_\tau^t d\tau'\int_\tau^{\tau'}d\tau''\;\psi(\tau-\tau'')\psi(\tau''-\tau')\psi(\tau'-t)\times\frac{\partial\hat{u}_\gamma(\mathbf{r}(\tau'),\tau')}{\partial r_\phi(\tau')}\frac{\partial\hat{u}_\phi(\mathbf{r}(\tau''),\tau'')}{\partial r_\beta(\tau'')}. \quad \text{(A5)}$$

It is possible to see that, to $O(\epsilon^2)$, this term does not contribute to the current J_α. In fact, including $\hat{R}^{(2)}_{\gamma\beta}$ in Eq. (12), the contraction $\partial_\phi\hat{u}_\gamma\partial_\beta\hat{u}_\phi\propto\epsilon\delta(\tau'-\tau'')$ would be killed by the factor $\psi(\tau''-\tau')$ from Eq. (A5). On the other hand, the contractions of the fields $\partial_\phi\hat{u}_\gamma$ and $\partial_\beta\hat{u}_\phi$, with the other factors in Eq. (12) would produce additional factors $\epsilon^{1/2}$, beyond those from the fields themselves, and the final result would be of higher order in ϵ.

APPENDIX B: CONTRIBUTIONS TO THE PROBABILITY CURRENT

The contribution from $\hat{R}^{(1)}$ in Eq. (12) is evaluated using the functional integration by part formula (8) and keeping only $\hat{R}^{(0)}$ in the resulting expansion for the response function. Indicating by J^R_α this contribution,

$$J^R_\alpha = 4\int_{-\infty}^0 d\tau \int_\tau^0 d\tau' \int d^D z \int d^D z' \times e^\tau \psi(\tau-\tau')\psi^2(\tau')\partial_{z_\beta}\hat{g}_{\gamma\eta}(\mathbf{z},\mathbf{z}')\bar{\partial}_\gamma\bar{\partial}_\eta\langle \hat{g}_{\alpha\beta}(\mathbf{r}(\tau)) \times \delta(\mathbf{r}(\tau')-\mathbf{z})\delta(\mathbf{r}(\tau')-\mathbf{z}')\delta(\mathbf{r}(0)-\bar{\mathbf{r}})\rangle. \tag{B1}$$

Notice that, due to the condition $\tau'>\tau$, the functional derivative acted only on $\delta(\mathbf{r}(0)-\bar{\mathbf{r}})$ and not on $\hat{g}_{\alpha\beta}(\mathbf{r}(\tau))$. To $O(\epsilon^2)$, one sets $\mathbf{r}(\tau')=\mathbf{r}(0)$ in Eq. (B1) and carrying out the integrals one obtains the result

$$J^R_\alpha = \tfrac{1}{3}\rho(\bar{\mathbf{r}})\bar{\partial}_\beta\bar{\partial}_\eta\hat{g}_{\alpha\gamma}(\bar{\mathbf{r}})\bar{\partial}_\gamma\hat{g}_{\beta\eta}(\bar{\mathbf{r}}) \tag{B2}$$

plus terms involving spatial derivatives of ρ.

The other contribution in Eq. (12) is

$$J^g_\alpha = -2\bar{\partial}_\gamma\int_{-\infty}^0 d\tau\, e^\tau\psi(\tau)\langle[\hat{g}_{\alpha\gamma}(\mathbf{r}(\tau))-\hat{g}_{\alpha\beta}(\mathbf{r}(0))]\delta(\mathbf{r}(0)-\bar{\mathbf{r}})\rangle, \tag{B3}$$

which can be seen to contain both quadratic and linear terms in $\Delta\mathbf{r}$, with $\Delta\mathbf{r}$ given in Eq. (14). Writing

$$\hat{g}_{\alpha\beta}(\mathbf{r}(\tau))-\hat{g}_{\alpha\beta}(\mathbf{r}(0)) = \hat{G}_{\alpha\beta\eta\phi}[\Delta r_\eta(\tau)\Delta r_\phi(\tau)+r_\eta(0)\Delta r_\phi(\tau)+r_\phi(0)\Delta r_\eta(\tau)],$$

with $\hat{G}_{\alpha\beta\eta\phi}=(1/2)\partial_\eta\partial_\phi\hat{g}_{\alpha\beta}(\mathbf{r})$, substituting into Eq. (B3), and using Eq. (14) allows us to calculate immediately the quadratic term

$$J^{g,2}_\alpha = -\tfrac{5}{6}\rho(\bar{\mathbf{r}})\bar{\partial}_\beta\bar{\partial}_\eta\hat{g}_{\alpha\gamma}(\bar{\mathbf{r}})\bar{\partial}_\gamma\hat{g}_{\beta\eta}(\bar{\mathbf{r}}) \tag{B4}$$

plus terms involving, again, spatial derivatives of ρ; the linear terms lead only to contributions involving derivatives of ρ. Combining Eqs. (B2) and (B4) leads to Eq. (16).

APPENDIX C: EVALUATION OF NONERGODIC TERMS

As with Eq. (7), the calculation of conditional averages like $\langle f_n[\hat{\mathbf{u}}]|\mathbf{r}(0)=\bar{\mathbf{r}}\rangle$, with $f_n[\bar{\mathbf{u}}]$ given in Eqs. (21)–(23), is carried on by application of the functional integration by part formula (8). In the case of $f_1[\hat{\mathbf{u}}]$, one has to lowest order in ϵ, from Eqs. (8) and (10),

$$\langle\bar{\partial}_\alpha\hat{u}_\alpha(\bar{\mathbf{r}},\tau)\delta(\mathbf{r}(0)-\bar{\mathbf{r}})\rangle = 2[\bar{\partial}_\beta\langle\delta(\mathbf{r}(0)-\bar{\mathbf{r}})\bar{\partial}_\alpha\hat{g}_{\alpha\beta}(\bar{\mathbf{r}},\mathbf{r}(\tau))\rangle - \langle\delta(\mathbf{r}(0)-\bar{\mathbf{r}})\bar{\partial}_\alpha\bar{\partial}_\beta\hat{g}_{\alpha\beta}(\bar{\mathbf{r}},\mathbf{r}(\tau))\rangle]. \tag{C1}$$

From the definition $2\hat{g}_{\alpha\beta}(\mathbf{r},\mathbf{r}')\delta(t-t')=\langle\hat{u}_\alpha(\mathbf{r},t)\hat{u}_\beta(\mathbf{r}',t')\rangle$, with $\hat{u}(\mathbf{r},t)=\mathbf{u}(\mathbf{x}+\mathbf{r},t)-\mathbf{u}(\mathbf{x},t)$, it is easy to see that $\hat{g}_{\alpha\beta}(\mathbf{r},\mathbf{r}')=(1/2)[\hat{g}_{\alpha\beta}(\mathbf{r})+\hat{g}_{\alpha\beta}(\mathbf{r}')-\hat{g}_{\alpha\beta}(\mathbf{r}-\mathbf{r}')]$, and therefore

$$\partial_{r_\alpha}\hat{g}_{\alpha\beta}(\mathbf{r},\mathbf{r}')|_{\mathbf{r}'=\mathbf{r}} = (1/2)\partial_{r_\alpha}\hat{g}_{\alpha\beta}(\mathbf{r}). \tag{C2}$$

Substituting into Eq. (C1) with the approximation (valid to the order considered) $\hat{g}_{\alpha\beta}(\bar{\mathbf{r}},\mathbf{r}(\tau))\simeq\hat{g}_{\alpha\beta}(\bar{\mathbf{r}})$ leads immediately to Eq. (24). The calculation of $\langle f_3[\hat{\mathbf{u}}]|\mathbf{r}(0)=\bar{\mathbf{r}}\rangle$, which leads to Eq. (27), is completely analogous.

The calculation of $\langle f_2[\hat{\mathbf{u}}]|\mathbf{r}(0)=\bar{\mathbf{r}}\rangle$ is slightly more involved and requires considering the correction terms in the response function [Eq. (11)]. The starting point is the equation

$$\langle\hat{u}_\alpha(\bar{\mathbf{r}},\tau)\hat{u}_\beta(\bar{\mathbf{r}},\tau')\delta(\mathbf{r}(0)-\bar{\mathbf{r}})\rangle = 4\psi(\tau)\psi(\tau')\langle\hat{g}_{\alpha\gamma}(\bar{\mathbf{r}},\mathbf{r}(\tau))\hat{g}_{\beta\eta}(\bar{\mathbf{r}},\mathbf{r}(\tau'))\bar{\partial}_\gamma\bar{\partial}_\eta\delta(\mathbf{r}(0)-\bar{\mathbf{r}})\rangle - 2\langle\hat{u}_\alpha(\mathbf{r}(\tau),\tau)\hat{g}_{\beta\gamma}(\bar{\mathbf{r}},\mathbf{r}(\tau')) \times R^{(1)}_{\gamma\phi}(0;\mathbf{r}(\tau'),\tau')\bar{\partial}_\phi\delta(\mathbf{r}(0)-\bar{\mathbf{r}})\rangle. \tag{C3}$$

For $\tau>\tau'$, the product $\hat{u}_\alpha(\mathbf{r}(\tau),\tau)R^{(1)}_{\eta\phi}(0,\mathbf{r}(\tau'),\tau')$ in the second term to RHS of Eq. (C3), from Eq. (11), leads to a factor $2\psi(\tau'-\tau)\psi(\tau)\partial_{r_\gamma(\tau)}\hat{g}_{\alpha\eta}(\bar{\mathbf{r}},\mathbf{r}(\tau))=2\psi(\tau'-\tau)\psi(\tau)\bar{\partial}_\gamma\hat{g}_{\alpha\eta}(\bar{\mathbf{r}},\mathbf{r}(\tau))$. Substituting into Eq. (C3), approximating $\hat{g}_{\alpha\beta}(\bar{\mathbf{r}},\mathbf{r}(\tau))\simeq\hat{g}_{\alpha\beta}(\bar{\mathbf{r}},\mathbf{r}(\tau'))\simeq\hat{g}_{\alpha\beta}(\bar{\mathbf{r}})$, and using Eq. (C2) leads, after little algebra, to Eq. (26).

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